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A GROUP THEORETIC METHOD FOR ELASTIC DIELECTRICS

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Abstract—Schur's lemma is applied to linear constitutive equations of elastic dielectrics which remain invariant under a group of symmetry transformations. The method of group representation theory is discussed in detail to generate constitutive equations for alpha-quartz which belongs to the $D_3(32)$ symmetry group. The constitutive equations thus constructed agree with those obtained by Mindlin and Toupin[1].

1. INTRODUCTION

The problem of constructing explicit constitutive equations which remain invariant under a group of symmetry transformations has been the subject of many investigations in recent years. To incorporate symmetry restrictions in non-linear constitutive equations, it was customary to follow the method of Voigt[2] where one starts with polynomial expansions and then investigates the restrictions which the material symmetry imposes on the constant coefficients in such expansions. This method is cumbersome and increases in complexity with the increase in degree of the terms. More sophisticated procedures have been developed by Fieschi[3], Fumi[4], Callen[5] and Nye[6] which determine the exact number of non-zero independent and dependent coefficients in the constitutive equations, as restricted by symmetry. Recently, a systematic and direct method of group representation theory has been developed by Smith and Kiral[7, 8] to incorporate symmetry restrictions in constitutive theory. Applications of methods of finite groups and symmetry to problems of mechanics and constitutive theory are discussed in [9, 10].

In this paper, the method of group representation theory and Schur's lemma are employed[7] to construct constitutive equations for α-quartz, which belongs to the crystallographic point group $D_3(32)$. The linear constitutive equations for the elastic dielectric with arbitrary symmetry are written in matrix form containing six components of each of the symmetric stress and strain tensors $(\bar{\sigma}, \bar{S})$, three components of each of the electric and polarization vectors $({}_L\bar{E}, \bar{P})$, and nine components of each of the electric and polarization gradient tensors $(\bar{\varepsilon}, \bar{\Pi})$, involving 171 independent constants. The symmetry group for α-quartz consists of six elements which are 3×3 matrices. Carrier spaces associated with the irreducible representations of the group are constructed and the number of independent constants reduced to 34.

The set of constitutive equations thus obtained, agree with those derived by Mindlin and Toupin[1].

2. LINEAR CONSTITUTIVE EQUATIONS OF ELASTIC DIELECTRICS

Let a homogeneous linear elastic dielectric continuum with the contribution of polarization gradient taken into account and bounded by a surface S, occupy a region V in a rectangular Cartesian coordinate system.

The general quadratic expression for the strain energy density of deformation and polarization is given by

$$W^L = \frac{1}{2}c_{klij}S_{ij}S_{kl} + \frac{1}{2}a_{ij}P_iP_j + \frac{1}{2}b_{ijkl}\Pi_{ij}\Pi_{kl} + f_{ijk}S_{jk}P_i + j_{ijk}P_i\Pi_{jk} + d_{ijkl}\Pi_{ij}S_{kl} \tag{2.1}$$

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where S_{ij} are the components of the symmetric strain tensor, P_i are components of the polarization vector and $\Pi_{ij} = P_{j,i}$ are the components of the polarization gradient tensor. Surface energy effects are assumed to be negligible.

The constitutive equations for the components of stress tensor σ_{ij}, the local electric vector, ${}_LE_i$, and the electric tensor, ϵ_{ij}, are given by [1]

$$\sigma_{ij} = \frac{\partial W^L}{\partial S_{ij}} = c_{ijkl}S_{kl} + f_{kij}P_k + d_{klij}\Pi_{kl} \tag{2.2}$$

$$-{}_LE_i = \frac{\partial W^L}{\partial P_i} = f_{ikl}S_{kl} + a_{ik}P_k + j_{ikl}\Pi_{kl} \tag{2.3}$$

$$\epsilon_{ij} = \frac{\partial W^L}{\partial \Pi_{ij}} = d_{ijkl}S_{kl} + j_{kij}P_k + b_{ijkl}\Pi_{kl}. \tag{2.4}$$

Next, introduce the following abbreviated matrix notation,

$$\begin{aligned}
&\bar{\sigma}^T = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12}], \qquad {}_L\bar{E}^T = [E_1, E_2, E_3]\\
&\bar{S}^T = [S_{11}, S_{22}, S_{33}, S_{23}, S_{31}, S_{12}], \qquad \bar{P}^T = [P_1, P_2, P_3]\\
&\bar{\epsilon}^T = [\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{23}, \epsilon_{32}, \epsilon_{31}, \epsilon_{13}, \epsilon_{12}, \epsilon_{21}]\\
&\bar{\Pi}^T = [\Pi_{11}, \Pi_{22}, \Pi_{33}, \Pi_{23}, \Pi_{32}, \Pi_{31}, \Pi_{13}, \Pi_{12}, \Pi_{21}]
\end{aligned} \tag{2.5}$$

where the superscript T denotes the transpose of the column vector. The scheme in which a pair of indices, ij, or kl, is replaced by a single index is indicated in Tables 1–3.

The system of constitutive equations, can now be written in the matrix form as

$$\underset{(6\times 1)}{\bar{\sigma}} = \underset{(6\times 6)(6\times 1)}{\bar{\bar{c}}\cdot\bar{S}} + \underset{(6\times 3)(3\times 1)}{\bar{\bar{f}}^t\cdot\bar{P}} + \underset{(6\times 9)(9\times 1)}{\bar{\bar{d}}^t\cdot\bar{\Pi}}, \tag{2.6}$$

$$\underset{(3\times 1)}{-{}_L\bar{E}} = \underset{(3\times 6)(6\times 1)}{\bar{\bar{f}}\cdot\bar{S}} + \underset{(3\times 3)(3\times 1)}{\bar{\bar{a}}\cdot\bar{P}} + \underset{(3\times 9)(9\times 1)}{\bar{\bar{j}}\cdot\bar{\Pi}}, \tag{2.7}$$

Table 1. Indexing scheme for c_{ijkl}, g_{mij}

$(ij), (kl)$	11	22	33	23,32	31,13	12,21
	1	2	3	4	5	6

, $m = 1$–3

Table 2. Indexing scheme for b_{ijkl}, j_{mkl}

$(ij), (kl)$	11	22	33	23	32	31	13	12	21
	1	2	3	4	5	6	7	8	9

, $m = 1$–6

Table 3. Indexing scheme for d_{klij}

(kl)	11 1	22 2	33 3	23 4	32 5	31 6	13 7	12 8	21 9
(ij)	11	22	33	23,32	31,13	12,21			
	1	2	3	4	5	6			

$$\underset{(9\times 1)}{\bar{\epsilon}} = \underset{(9\times 6)(6\times 1)}{\bar{\bar{d}}\cdot\bar{S}} + \underset{(9\times 3)(3\times 1)}{\bar{\bar{f}}^t\cdot\bar{P}} + \underset{(9\times 9)(9\times 1)}{\bar{\bar{b}}\cdot\bar{\Pi}}, \tag{2.8}$$

where the numbers in parentheses below the matrix indicate the order of the matrix. For an elastic dielectric with arbitrary symmetry, the total number of independent material constants is 171. A group of symmetry transformations under which the constitutive equations remain invariant leave some constants mutually dependent and make others vanish.

3. ALPHA-QUARTZ—THE SYMMETRY GROUP $D_3(32)$

Of all the elastic dielectrics, α-quartz is an optically active and birefringent material. It belongs to the $D_3(32)$ crystallographic group. The matrices comprising the symmetry group, Γ, of this crystal class are given by

$$\bar{\bar{A}}_1 \Rightarrow \begin{vmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{vmatrix} \qquad \bar{\bar{A}}_2 \Rightarrow \begin{vmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & \cdot \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & \cdot \\ \cdot & \cdot & 1 \end{vmatrix}$$

$$\bar{\bar{A}}_3 \Rightarrow \begin{vmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & \cdot \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & \cdot \\ \cdot & \cdot & 1 \end{vmatrix} \qquad \bar{\bar{A}}_4 \Rightarrow \begin{vmatrix} 1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \end{vmatrix}$$

$$\bar{\bar{A}}_5 \Rightarrow \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & \cdot \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & \cdot \\ \cdot & \cdot & -1 \end{bmatrix} \qquad \bar{\bar{A}}_6 \Rightarrow \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & \cdot \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & \cdot \\ \cdot & \cdot & -1 \end{bmatrix}$$

where . denotes a zero component.

There are three inequivalent irreducible representations[8] $D_1(A_\alpha)$, $D_2(A_\alpha)$ and $D_3(A_\alpha)$, $\alpha = 1$–6 associated with the group $\Gamma = \{\bar{\bar{A}}_1, \bar{\bar{A}}_2, \bar{\bar{A}}_3, \bar{\bar{A}}_4, \bar{\bar{A}}_5, \bar{\bar{A}}_6\}$ which are of degree one and two and are listed in Table 4.

The determination of the form of the constitutive equations (2.2)–(2.4), which are invariant under the group Γ, becomes apparent when $\bar{S}$, $\bar{P}$ and $\bar{\Pi}$ are decomposed into sets which form the carrier spaces of irreducible representation of Γ, $D_i(A_\alpha)$; $i = 1, 2, 3$ and $\alpha = 1, \ldots, 6$.

The six independent components $(S_{11}, S_{22}, S_{33}, S_{23}, S_{31}, S_{12})$ of the symmetric tensor $\bar{S}_{ij}$, the three independent components (P_1, P_2, P_3) of the vector $\bar{P}$, and the nine components $(\Pi_{11}, \Pi_{22}, \Pi_{33}, \Pi_{23}, \Pi_{32}, \Pi_{31}, \Pi_{13}, \Pi_{12}, \Pi_{21})$ of the arbitrary tensor Π_{ij}, can be split into the sets[7]

(1) $S_{33}^{(1)}$, $(S_{11}+S_{22})^{(1)}$, $\Pi_{33}^{(1)}$, $(\Pi_{11}+\Pi_{22})^{(1)}$;

(2) $P_3^{(2)}$, $(\Pi_{12}-\Pi_{21})^{(2)}$;

Table 4. Irreducible representations of $D_3(32)$

	$\bar{\bar{A}}_1$	$\bar{\bar{A}}_2$	$\bar{\bar{A}}_3$	$\bar{\bar{A}}_4$	$\bar{\bar{A}}_5$	$\bar{\bar{A}}_6$
$D_1(A_\alpha)$	1	1	1	1	1	1
$D_2(A_\alpha)$	1	1	1	−1	−1	−1
$D_3(A_\alpha)$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,	$\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$,	$\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$,	$\begin{bmatrix} -1 & 0 \\ 0 & +1 \end{bmatrix}$,	$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$,	$\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

(3) $\{S_{13}, S_{23}\}^{(3)}, \{2S_{12}, S_{11}-S_{22}\}^{(3)}, \{P_2, -P_1\}^{(3)},$
$\{\Pi_{13}, \Pi_{23}\}^{(3)}, \{\Pi_{31}, \Pi_{32}\}^{(3)}, \{\Pi_{12}+\Pi_{21}, \Pi_{11}-\Pi_{22}\}^{(3)};$ (3.3)

the transformation properties of which are defined by D_1, D_2, D_3. The new sets of quantities, $\bar{S}^*$, $\bar{P}^*$ and $\bar{\Pi}^*$ (which are the carrier spaces of the irreducible representations) are related to the original sets by the following transformations

$$\bar{S}^* = \bar{\bar{Q}}_{(S)} \cdot \bar{S}, \qquad \bar{P}^* = \bar{\bar{Q}}_{(P)} \cdot \bar{P}, \qquad \bar{\Pi}^* = \bar{\bar{Q}}_{(\Pi)} \cdot \bar{\Pi} \tag{3.2}$$

where

$$\bar{S}^{*t} = [S_{33}^{(1)}, (S_{11}+S_{22})^{(1)}, \{S_{13}, S_{23}\}^{(3)}, \{2S_{12}, (S_{11}-S_{22})\}^{(3)}], \tag{3.3a}$$

$$\bar{P}^{*t} = [P_3^{(2)}, \{P_2, -P_1\}^{(3)}], \tag{3.3b}$$

$$\begin{aligned}\bar{\Pi}^{*t} = {} & [\Pi_{33}^{(1)}, (\Pi_{11}+\Pi_{22})^{(1)}, (\Pi_{12}-\Pi_{21})^{(2)}, \{\Pi_{13}, \Pi_{23}\}^{(3)}, \\ & \times \{\Pi_{31}, \Pi_{32}\}^{(3)}, \{(\Pi_{12}+\Pi_{21}), (\Pi_{11}-\Pi_{22})\}^{(3)}],\end{aligned} \tag{3.3c}$$

and where the matrices for $\bar{\bar{Q}}_{(S)}$, $\bar{\bar{Q}}_{(P)}$, $\bar{\bar{Q}}_{(\Pi)}$ as well as for their inverses are given in Appendix A.

4. REDUCTION BY SCHUR'S LEMMA

Multiplying the constitutive equations (2.6)–(2.8) by $\bar{\bar{Q}}_{(S)}$, $\bar{\bar{Q}}_{(P)}$ and $\bar{\bar{Q}}_{(\Pi)}$, respectively, one can rewrite the system in the following form

$$\bar{\sigma}^* = \bar{\bar{c}}^* \cdot \bar{S}^* + \bar{\bar{f}}^{*t} \cdot \bar{P}^* + \bar{\bar{d}}^{*t} \cdot \bar{\Pi}^* \tag{4.1}$$

$$-{}_L\bar{E}^* = \bar{\bar{f}}^* \cdot \bar{S}^* + \bar{\bar{a}}^* \cdot \bar{P}^* + \bar{\bar{j}}^* \cdot \bar{\Pi}^* \tag{4.2}$$

$$\bar{\epsilon}^* = \bar{\bar{d}}^* \cdot \bar{S}^* + \bar{\bar{j}}^{*t} \cdot \bar{P}^* + \bar{\bar{b}}^* \cdot \bar{\Pi}^* \tag{4.3}$$

where

$$\bar{\sigma}^* = \bar{\bar{Q}}_{(S)} \cdot \bar{\sigma}, \qquad \bar{E}^* = \bar{\bar{Q}}_{(P)} \cdot \bar{P}, \qquad \bar{\epsilon}^* = \bar{\bar{Q}}_{(\Pi)} \cdot \bar{\Pi}, \tag{4.4}$$

$\bar{S}^*$, $\bar{P}^*$, $\bar{\Pi}^*$ are defined by (3.2), and the coefficient matrices are given by

$$\begin{bmatrix} \bar{\bar{c}}^* & \bar{\bar{f}}^{*t} & \bar{\bar{d}}^{*t} \\ \bar{\bar{f}}^* & \bar{\bar{a}}^* & \bar{\bar{j}}^* \\ \bar{\bar{d}}^* & \bar{\bar{j}}^{*t} & \bar{\bar{b}}^* \end{bmatrix} = \begin{bmatrix} \bar{\bar{Q}}_{(S)} & \cdot & \cdot \\ \cdot & \bar{\bar{Q}}_{(P)} & \cdot \\ \cdot & \cdot & \bar{\bar{Q}}_{(\Pi)} \end{bmatrix} \begin{bmatrix} \bar{\bar{c}} & \bar{\bar{f}}^t & \bar{\bar{d}}^t \\ \bar{\bar{f}} & \bar{\bar{a}} & \bar{\bar{j}} \\ \bar{\bar{d}} & \bar{\bar{j}}^t & \bar{\bar{b}} \end{bmatrix} \begin{bmatrix} \bar{\bar{Q}}_{(S)}^{-1} & \cdot & \cdot \\ \cdot & \bar{\bar{Q}}_{(P)}^{-1} & \cdot \\ \cdot & \cdot & \bar{\bar{Q}}_{(\Pi)}^{-1} \end{bmatrix}$$

Making use of the matrices (A1)–(A4) and Schur's Lemma[7], the coefficient matrices assume the forms shown in Appendix B. The following observations can be made from the matrices listed in Appendix B.

(i) The matrices $\bar{\bar{a}}$, $\bar{\bar{b}}$ and $\bar{\bar{c}}$ are symmetric.

(ii) In all starred (*) coefficient matrices $\diagdown$ represents a constant times $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ and is associated with the elements of the corresponding column matrix (in the constitutive equations) which form the carrier space of the irreducible representation D_3.

(iii) In the matrix $\bar{\bar{c}}^*$, the elements c_{11}^*, c_{12}^*, c_{21}^* and c_{22}^* are associated with the elements $S_{33}^{(1)}$, $(S_{11}+S_{22})^{(1)}$ of the column vector $\bar{S}^*$ which form the carrier space of the irreducible representation D_1.

(iv) In the matrix $\bar{\bar{a}}^*$, the element a_{11}^* is associated with the element $P_3^{(2)}$ of the column vector $\bar{P}^*$ which form the carrier space of the irreducible representation D_2.

(v) In the matrix $\bar{\bar{j}}^*$, the element j_{13}^* is associated with the element $\Pi_{12}^{(2)} - \Pi_{21}^{(2)}$ of the column vector $\bar{\Pi}^*$ and j_{31}^* is associated with the element $P_3^{(2)}$ of $\bar{P}^*$ which are both carrier spaces of irreducible representation D_2.

(vi) In the matrix $\bar{\bar{d}}^*$, d_{11}^*, d_{21}^*, d_{12}^*, d_{22}^* are associated with the elements $\Pi_{33}^{(1)}$, $(\Pi_{11}+\Pi_{22})^{(1)}$ of the column vector $\bar{\Pi}^*$ or the elements $S_{33}^{(1)}$, $(S_{11}+S_{22})^{(1)}$ of the column vector $\bar{S}^*$ which are both the carrier space of the irreducible representation D_1.

(vii) In the matrix $\bar{\bar{b}}^*$, the elements b_{11}^*, b_{12}^*, b_{21}^*, b_{22}^* are associated with the elements $\Pi_{33}^{(1)}$, $(\Pi_{11}+\Pi_{22})^{(1)}$ of the column vector $\bar{\Pi}^*$ which are the carrier space of irreducible representation D_1; b_{33}^* is associated with the elements $(\Pi_{12}-\Pi_{21})^{(2)}$ which is the carrier space of irreducible representation D_2.

A comparison of corresponding elements leads to a system of algebraic equations listed in Appendix C, the solution of which results in the following restrictions on the elastic and dielectric constants.

(a) *For matrix* $\bar{\bar{c}}$

$$c_{5j}=c_{j5}=c_{6j}=c_{j6}=0 \qquad (j=1,2,3,4)$$
$$c_{43}=c_{34}=0,$$

while c_{11}, c_{12}, c_{13}, c_{14}, c_{33}, c_{44} are distinct non-zero constants, and c_{22}, c_{23}, c_{24}, c_{55}, c_{56}, c_{66} are dependent non-zero constants, given by

$$c_{22}=c_{11}, \qquad c_{23}=c_{13}, \qquad c_{24}=-c_{14},$$
$$c_{55}=c_{44}, \qquad c_{56}=c_{14}, \qquad c_{66}=(c_{11}-c_{12}).$$

Thus the number of independent constants in $\bar{\bar{c}}$ is 6.

(b) *For matrix* $\bar{\bar{f}}$

$$f_{2j}=f_{3j}=0 \qquad (j=1,2,3,4)$$
$$f_{1j}=f_{3j}=0 \qquad (j=5,6)$$

while f_{11} and f_{14} are non-zero distinct constants, and f_{12}, f_{25}, f_{26} are dependent non-zero constants given as

$$f_{12}=-f_{11}, \qquad f_{25}=-f_{14}, \qquad f_{26}=-f_{11}.$$

Thus the number of independent constants in the matrix $\bar{\bar{f}}$ is 2.

(c) *For matrix* $\bar{\bar{a}}$

$$a_{ij}=a_{11}(\delta_{1i}\delta_{1j}+\delta_{2i}\delta_{2j})+a_{33}\delta_{i3}\delta_{j3}$$

where δ_{ij} is the Kronecker delta. The number of independent constants in the matrix $\bar{\bar{a}}$ is 2.

(d) *For matrix* $\bar{\bar{j}}$

$$j_{1k}=0 \quad (k=6\text{–}9), \qquad j_{13}=j_{36}=j_{37}=0$$
$$j_{2k}=j_{3k}=0 \qquad (k=1\text{–}5)$$

where j_{11}, j_{14}, j_{15}, j_{38} are distinct non-zero constants and j_{12}, j_{26}, j_{27}, j_{28}, j_{29}, j_{39} are dependent non-zero constants given by

$$j_{12}=-j_{11}, \qquad j_{26}=-j_{15}, \qquad j_{27}=-j_{14}, \qquad j_{28}=-j_{11},$$
$$j_{29}=-j_{11}, \qquad j_{39}=-j_{38}.$$

Thus the number of independent constants in the matrix $\bar{\bar{j}}$ is 4.

(e) *For matrix* $\bar{d}$

$$d_{k5} = d_{k6} = 0 \qquad (k = 1\text{–}5)$$
$$d_{6k} = d_{7k} = d_{8k} = d_{9k} = 0 \qquad (k = 1\text{–}4)$$
$$d_{53} = d_{43} = d_{34} = 0$$

while d_{11}, d_{12}, d_{13}, d_{14}, d_{31}, d_{33}, d_{41}, d_{44}, d_{51}, d_{54} are distinct non-zero constants and d_{21}, d_{22}, d_{23}, d_{24}, d_{32}, d_{42}, d_{52}, d_{65}, d_{66}, d_{75}, d_{76}, d_{85}, d_{86}, d_{95} and d_{96} are dependent non-zero constants defined as

$$d_{21} = d_{12}, \quad d_{22} = d_{11}, \quad d_{23} = d_{13}, \quad d_{24} = -d_{14}$$
$$d_{32} = d_{31}, \quad d_{42} = -d_{41}, \quad d_{52} = -d_{51}, \quad d_{65} = d_{54}, \quad d_{66} = d_{51}$$
$$d_{75} = d_{44}, \quad d_{76} = d_{41}, \quad d_{85} = d_{14}, \quad d_{86} = \frac{1}{2}(d_{11} - d_{12})$$
$$d_{95} = d_{14}, \quad d_{96} = \frac{1}{2}(d_{11} - d_{12}).$$

Thus the number of independent constants in matrix $\bar{d}$ is 10.

(f) *For matrix* $\bar{b}$ *(symmetric)*

$$b_{ij} = 0 \qquad (i = 1\text{–}5, j = 6\text{–}9) \quad \text{and} \quad (i = 6\text{–}9, j = 1\text{–}5)$$
$$b_{34} = b_{35} = b_{43} = b_{53} = 0$$

while b_{11}, b_{12}, b_{13}, b_{14}, b_{15}, b_{44}, b_{45}, b_{55}, b_{88}, b_{89} are distinct independent non-zero constants and b_{22}, b_{23}, b_{24}, b_{25}, b_{42}, b_{52}, b_{66}, b_{67}, b_{68}, b_{69}, b_{77}, b_{78}, b_{79}, b_{99} are dependent non-zero constants given by

$$b_{22} = b_{11}, \quad b_{23} = b_{13}, \quad b_{24} = -b_{14}, \quad b_{25} = -b_{15}$$
$$b_{42} = -b_{14}, \quad b_{52} = -b_{51}, \quad b_{66} = b_{55}, \quad b_{67} = b_{45}, \quad b_{68} = b_{51},$$
$$b_{69} = b_{51}$$
$$b_{76} = b_{45}, \quad b_{77} = b_{44}, \quad b_{78} = b_{14}, \quad b_{79} = b_{14}$$
$$b_{86} = b_{15}, \quad b_{87} = b_{14}, \quad b_{96} = b_{15}, \quad b_{97} = b_{14}, \quad b_{99} = b_{88}.$$

Thus the number of independent constants in matrix $\bar{b}$ is 10.

The 171 $(21+18+6+27+54+45)$ independent constants of an elastic dielectric with arbitrary symmetry are thus reduced to $34(6+2+2+4+10+10)$ for an alpha quartz belonging to the $D_3(32)$ symmetry group.

The system of constitutive equations (2.6)–(2.8) for the α-quartz assumes the form

	S_{11}	S_{22}	S_{33}	S_{23}	S_{31}	S_{12}	P_1	P_2	P_3	Π_{11}	Π_{22}	Π_{33}	Π_{23}	Π_{32}	Π_{31}	Π_{13}	Π_{12}	Π_{21}
σ_{11}	c_{11}	c_{12}	c_{13}	c_{14}	.	.	f_{11}	.	.	d_{11}	d_{12}	d_{31}	d_{41}	d_{51}	.	.	.	.
σ_{22}	c_{12}	c_{11}	c_{13}	$-c_{14}$	.	.	f_{11}	.	.	d_{12}	d_{11}	d_{31}	$-d_{41}$	$-d_{51}$	.	.	.	
σ_{33}	c_{13}	c_{13}	c_{33}	.	.	.	.	.	.	d_{13}	d_{13}	d_{33}	.	.	.	.	.	.
c_{23}	c_{14}	$-c_{14}$	.	c_{44}	.	.	f_{14}	.	.	d_{14}	$-d_{14}$	.	d_{44}	d_{54}	.	.	.	.
σ_{31}	.	.	.	.	c_{44}	c_{14}	.	$-f_{14}$	.	.	.	.	.	.	d_{54}	d_{44}	d_{14}	d_{14}
σ_{12}	.	.	.	.	c_{14}	c_{66}	.	$-f_{11}$	.	.	.	.	.	.	d_{51}	d_{41}	d_{86}	d_{86}
$-{}_LE_1$	f_{11}	$-f_{11}$	.	f_{14}	.	.	a_{11}	.	.	j_{11}	$-j_{11}$	.	j_{14}	j_{15}	.	.	.	.
$-{}_LE_2$	.	.	.	.	$-f_{14}$	$-f_{11}$	.	a_{11}	.	.	.	.	.	.	$-j_{15}$	$-j_{14}$	$-j_{11}$	$-j_{11}$
$-{}_LE_3$	.	.	.	.	.	.	.	.	a_{33}	.	.	.	.	.	.	.	j_{38}	$-j_{38}$
ϵ_{11}	d_{11}	d_{12}	d_{13}	d_{14}	.	.	j_{11}	.	.	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	.	.	.	.
ϵ_{22}	d_{12}	d_{11}	d_{13}	$-d_{14}$	.	.	$-j_{11}$	.	.	b_{12}	b_{11}	b_{13}	$-b_{14}$	$-b_{15}$	.	.	.	.
ϵ_{33}	d_{31}	d_{31}	d_{33}	.	.	.	.	.	.	b_{13}	b_{13}	b_{33}	.	.	.	.	.	.
ϵ_{23}	d_{41}	$-d_{41}$	.	d_{44}	.	.	j_{14}	.	.	b_{14}	$-b_{14}$	.	b_{44}	b_{45}	.	.	.	.
ϵ_{32}	d_{51}	$-d_{51}$	.	d_{54}	.	.	j_{15}	.	.	b_{15}	$-b_{15}$	.	b_{45}	b_{55}	.	.	.	.
ϵ_{31}	.	.	.	.	d_{54}	d_{51}	.	$-j_{15}$	.	.	.	.	.	.	b_{55}	b_{45}	b_{15}	b_{15}
ϵ_{13}	.	.	.	.	d_{44}	d_{41}	.	$-j_{14}$	.	.	.	.	.	.	b_{45}	b_{44}	b_{14}	b_{14}
ϵ_{12}	.	.	.	.	d_{14}	d_{86}	.	$-j_{11}$	j_{38}	.	.	.	.	.	b_{15}	b_{14}	b_{88}	b_{89}
ϵ_{21}	.	.	.	.	d_{14}	d_{86}	.	$-j_{11}$	$-j_{38}$	.	.	.	.	.	b_{15}	b_{14}	b_{89}	b_{88}

where

$$c_{66} = c_{11} - c_{22}, \qquad b_{88} + b_{99} = b_{11} - b_{22}, \qquad d_{86} = \frac{1}{2}(d_{11} - d_{12})$$

and which, with minor changes in notation, agree with those given by Mindlin and Toupin[1].

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APPENDIX A

The matrices $\bar{\dot{Q}}_{(S)}$, $\bar{\dot{Q}}_{(P)}$ and $\bar{\dot{Q}}_{(\Pi)}$ which appear in eqn (3.4), are given as

$$\bar{\dot{Q}}_{(S)} = \begin{bmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2 \\ 1 & -1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \qquad \bar{\dot{Q}}_{(P)} = \begin{bmatrix} \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \\ -1 & \cdot & \cdot \end{bmatrix}. \tag{A1}$$

$$Q_{(\Pi)} = \begin{bmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ 1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}. \tag{A2}$$

The inverse of these matrices $\bar{\dot{Q}}_{(S)}$, $\bar{\dot{Q}}_{(P)}$ and $\bar{\dot{Q}}_{(\Pi)}$ can be easily evaluated and are given by

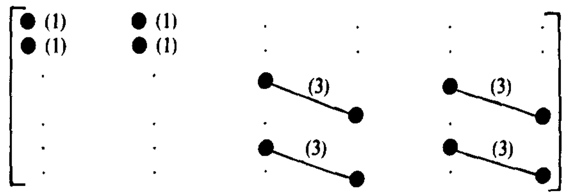
$$\bar{\dot{Q}}^{-1}_{(S)} = \begin{bmatrix} \cdot & 1/2 & \cdot & \cdot & \cdot & 1/2 \\ \cdot & 1/2 & \cdot & \cdot & \cdot & -1/2 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1/2 & \cdot \end{bmatrix} \qquad \bar{\dot{Q}}^{-1}_{(P)} = \begin{bmatrix} \cdot & \cdot & -1 \\ \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \end{bmatrix}. \tag{A3}$$

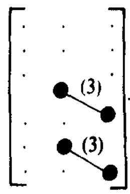
$$\bar{\dot{Q}}^{-1}_{(\Pi)} = \begin{bmatrix} \cdot & 1/2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1/2 \\ \cdot & 1/2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1/2 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1/2 & \cdot & \cdot & \cdot & \cdot & 1/2 & \cdot \\ \cdot & \cdot & -1/2 & \cdot & \cdot & \cdot & \cdot & 1/2 & \cdot \end{bmatrix}. \tag{A4}$$

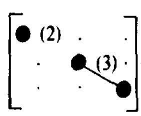
APPENDIX B

The matrices $\bar{c}^*$, $\bar{f}^{*t}$, $\bar{a}^*$, $\bar{g}^*$, $\bar{d}^{*t}$ and $\bar{b}^*$ listed in eqn (4.5) are evaluated using (A.1)–(A.4) and are given by

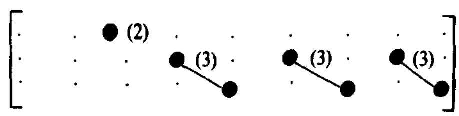
$$\bar{c}^* = \bar{Q}_{(S)}\bar{c}^t\bar{Q}_{(S)}^{-1} = \begin{bmatrix} c_{33} & \frac{c_{13}+c_{23}}{2} & c_{53} & c_{43} & \frac{1}{2}c_{63} & \frac{c_{13}-c_{23}}{2} \\ c_{31}+c_{32} & c_{12}+\frac{c_{11}+c_{22}}{2} & c_{51}+c_{52} & c_{41}+c_{42} & \frac{c_{61}+c_{62}}{2} & \frac{c_{11}-c_{22}}{2} \\ c_{35} & \frac{c_{15}+c_{25}}{2} & c_{55} & c_{45} & \frac{1}{2}c_{65} & \frac{c_{15}-c_{25}}{2} \\ c_{34} & \frac{c_{14}+c_{24}}{2} & c_{54} & c_{44} & \frac{1}{2}c_{64} & \frac{c_{14}-c_{24}}{2} \\ 2c_{36} & c_{12}+c_{26} & 2c_{56} & 2c_{46} & c_{66} & c_{16}-c_{26} \\ c_{31}-c_{32} & \frac{c_{11}-c_{22}}{2} & c_{51}-c_{52} & c_{41}-c_{42} & \frac{c_{61}-c_{62}}{2} & \frac{c_{11}+c_{22}}{2}-c_{12} \end{bmatrix}$$

=  . (B1)

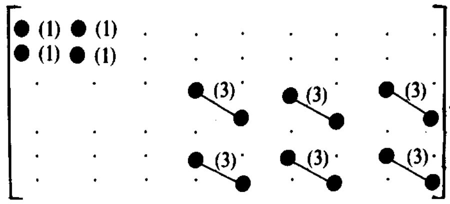
$$\bar{f}^{*t} = \bar{Q}_{(S)}\bar{f}^t\bar{Q}_{(P)}^{-1} = \begin{bmatrix} f_{33} & f_{23} & -f_{13} \\ f_{31}+f_{32} & f_{21}+f_{22} & -f_{11}-f_{12} \\ f_{35} & f_{25} & -f_{15} \\ f_{34} & f_{24} & -f_{14} \\ 2f_{36} & 2f_{26} & -2f_{16} \\ f_{31}-f_{32} & f_{21}-f_{22} & -f_{11}+f_{12} \end{bmatrix} =$$  . (B2)

$$\bar{a}^* = \bar{Q}_{(P)}\bar{a}\bar{Q}_{(P)}^{-1} = \begin{bmatrix} a_{33} & a_{32} & -a_{31} \\ a_{23} & a_{22} & -a_{21} \\ -a_{13} & -a_{12} & a_{11} \end{bmatrix} =$$  . (B3)

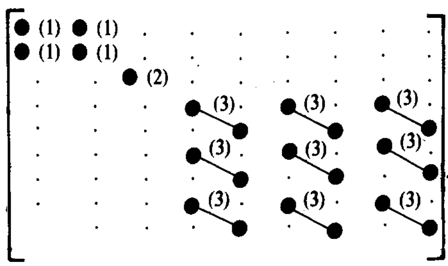
$$\bar{j}^* = \bar{Q}_{(P)}\bar{\bar{j}}\bar{Q}_{(\Pi)}^{-1} = \begin{bmatrix} j_{33} & \frac{1}{2}(j_{31}+j_{32}) & \frac{1}{2}(j_{38}+j_{39}) & j_{37} & j_{34} & j_{36} & j_{35} & \frac{1}{2}(j_{38}+j_{39}) & \frac{1}{2}(j_{31}-j_{32}) \\ j_{23} & \frac{1}{2}(j_{21}+j_{22}) & \frac{1}{2}(j_{28}-j_{29}) & j_{27} & j_{24} & j_{26} & j_{25} & \frac{1}{2}(j_{28}+j_{29}) & \frac{1}{2}(j_{21}-j_{22}) \\ -j_{13} & -\frac{1}{2}(j_{11}+j_{12}) & -\frac{1}{2}(j_{18}+j_{19}) & -j_{17} & -j_{14} & -j_{16} & -j_{15} & -\frac{1}{2}(j_{18}+j_{19}) & -\frac{1}{2}(j_{11}-j_{12}) \end{bmatrix}$$

=  . (B4)

$$\bar{d}^{*t} = \bar{Q}_{(S)}\bar{d}^t\bar{Q}_{(\Pi)}^{-1} = \begin{bmatrix} d_{33} & \frac{1}{2}(d_{13}+d_{23}) & \frac{1}{2}(d_{83}-d_{93}) & d_{73} & d_{43} & d_{63} & d_{53} & \frac{1}{2}(d_{83}+d_{93}) & \frac{1}{2}(d_{13}-d_{23}) \\ d_{31}+d_{32} & \frac{1}{2}(d_{11}+d_{12}+d_{21}+d_{22}) & \frac{1}{2}(d_{81}-d_{91}-d_{82}-d_{92}) & d_{71}+d_{72} & d_{42}+d_{42} & d_{61}+d_{62} & d_{51}+d_{52} & \frac{1}{2}(d_{81}+d_{91}+d_{82}+d_{92}) & \frac{1}{2}(d_{11}-d_{21}+d_{12}-d_{22}) \\ d_{35} & \frac{1}{2}(d_{15}+d_{25}) & \frac{1}{2}(d_{85}-d_{95}) & d_{75} & d_{45} & d_{65} & d_{55} & \frac{1}{2}(d_{85}+d_{95}) & \frac{1}{2}(d_{15}-d_{25}) \\ d_{34} & \frac{1}{2}(d_{14}+d_{24}) & \frac{1}{2}(d_{84}-d_{94}) & d_{74} & d_{44} & d_{64} & d_{54} & \frac{1}{2}(d_{84}+d_{94}) & \frac{1}{2}(d_{14}-d_{24}) \\ 2d_{36} & d_{16}+d_{26} & d_{86}-d_{96} & 2d_{76} & 2d_{46} & 2d_{66} & 2d_{56} & d_{86}+d_{96} & d_{16}-d_{26} \\ d_{31}-d_{32} & \frac{1}{2}(d_{11}+d_{21}-d_{12}-d_{22}) & \frac{1}{2}(d_{81}-d_{91}-d_{82}+d_{92}) & d_{71}-d_{72} & d_{41}-d_{42} & d_{61}-d_{62} & d_{51}-d_{52} & \frac{1}{2}(d_{81}+d_{91}-d_{82}-d_{92}) & \frac{1}{2}(d_{11}-d_{21}-d_{12}+d_{22}) \end{bmatrix}$$

$=$  (B5)

$$\ddot{b}^{*}=[\ddot{Q}_{(11)}\ddot{b}\ddot{Q}_{(11)}^{-1}]=\begin{bmatrix}
b_{33} & \frac{1}{2}(b_{31}+b_{32}) & \frac{1}{2}(b_{38}-b_{39}) & b_{37} & b_{34} & b_{36} & b_{35} & \frac{1}{2}(b_{38}+b_{39}) & \frac{1}{2}(b_{31}-b_{32})\\
b_{13}+b_{23} & \frac{1}{2}(b_{11}+b_{21}+b_{12}+b_{22}) & \frac{1}{2}(b_{18}+b_{28}-b_{19}-b_{29}) & b_{17}+b_{27} & b_{14}+b_{24} & b_{16}+b_{26} & b_{15}+b_{25} & \frac{1}{2}(b_{18}+b_{28}+b_{19}+b_{29}) & \frac{1}{2}(b_{11}+b_{21}-b_{12}-b_{22})\\
b_{83}-b_{93} & \frac{1}{2}(b_{81}-b_{91}+b_{82}-b_{92}) & \frac{1}{2}(b_{88}-b_{98}-b_{89}+b_{99}) & b_{87}-b_{97} & b_{84}-b_{94} & b_{86}-b_{96} & b_{85}-b_{95} & \frac{1}{2}(b_{88}-b_{98}+b_{89}-b_{99}) & \frac{1}{2}(b_{81}-b_{91}-b_{82}+b_{92})\\
b_{73} & \frac{1}{2}(b_{71}+b_{72}) & \frac{1}{2}(b_{78}-b_{79}) & b_{77} & b_{74} & b_{76} & b_{75} & \frac{1}{2}(b_{78}+b_{79}) & \frac{1}{2}(b_{71}-b_{72})\\
b_{43} & \frac{1}{2}(b_{41}+b_{42}) & \frac{1}{2}(b_{48}-b_{49}) & b_{47} & b_{44} & b_{46} & b_{45} & \frac{1}{2}(b_{48}+b_{49}) & \frac{1}{2}(b_{41}-b_{42})\\
b_{63} & \frac{1}{2}(b_{61}-b_{62}) & \frac{1}{2}(b_{68}-b_{69}) & b_{67} & b_{64} & b_{66} & b_{65} & \frac{1}{2}(b_{68}+b_{69}) & \frac{1}{2}(b_{61}-b_{62})\\
b_{53} & \frac{1}{2}(b_{51}+b_{52}) & \frac{1}{2}(b_{58}-b_{59}) & b_{57} & b_{54} & b_{56} & b_{55} & \frac{1}{2}(b_{58}+b_{59}) & \frac{1}{2}(b_{51}-b_{52})\\
b_{83}+b_{93} & \frac{1}{2}(b_{81}+b_{91}+b_{82}+b_{92}) & \frac{1}{2}(b_{88}+b_{98}-b_{89}-b_{99}) & b_{87}+b_{97} & b_{84}+b_{94} & b_{86}+b_{96} & b_{85}+b_{95} & \frac{1}{2}(b_{88}+b_{98}+b_{89}+b_{99}) & \frac{1}{2}(b_{81}+b_{91}-b_{82}-b_{92})\\
b_{13}-b_{23} & \frac{1}{2}(b_{11}-b_{21}+b_{12}-b_{22}) & \frac{1}{2}(b_{18}-b_{28}-b_{19}+b_{29}) & b_{17}-b_{27} & b_{14}-b_{24} & b_{16}-b_{26} & b_{15}-b_{25} & \frac{1}{2}(b_{18}-b_{28}+b_{19}-b_{29}) & \frac{1}{2}(b_{11}-b_{21}-b_{12}+b_{22})
\end{bmatrix}$$

$=$  (B6)

where ., ●, ●——● denote respectively, the zero component, non-zero components and non-zero equal components, respectively.

The value of the elements in the coefficient matrices are decided by Schur's Lemma[7] and by comparing their elements on two sides of the new matrix equation associated with the same irreducible representation.

APPENDIX C

Systems of algebraic equations for the constant elements of the matrices $\ddot{c}$, $\ddot{f}$, $\ddot{a}$, $\ddot{j}$, $\ddot{d}$ and $\ddot{b}$ are obtained from eqns (B1)–(B6). Details are given here for the equations for matrix $\ddot{c}$ only [eqn (B1)].

$c_{33}\neq 0,\quad c_{13}+c_{23}\neq 0,\quad c_{53}=0,\quad c_{43}=0,\quad c_{63}=0,\quad c_{13}=c_{23}$

$c_{31}+c_{32}\neq 0,\quad c_{12}+\dfrac{c_{11}+c_{22}}{2}\neq 0,\quad c_{41}+c_{42}=0,\quad c_{61}+c_{62}=0,$

$c_{11}=c_{22},\quad c_{51}+c_{52}=0$

$c_{35}=0,\quad c_{15}+c_{25}=0,\quad c_{55}=c_{44},\quad c_{45}=0,\quad c_{65}=c_{14}-c_{24},$

$c_{15}=c_{25}$

$c_{34}=0,\quad c_{14}+c_{24}=0,\quad c_{54}=0,\quad c_{64}=0$

$c_{36}=0,\quad c_{16}+c_{26}=0,\quad 2c_{56}=c_{41}-c_{42},\quad c_{46}=0,$

$c_{66}=\dfrac{c_{11}+c_{22}}{2}-c_{12},\quad c_{16}=c_{26}$

$c_{31}=c_{32},\quad c_{11}=c_{22},\quad c_{51}=c_{52},\quad c_{61}=c_{62}.$

Equations for the remaining matrices can be obtained analogously.